

Applications of quantum cryptographic switch: Various tasks related to controlled quantum communication can be performed using Bell states and permutation of particles

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Abstract

Recently, several aspects of controlled quantum communication (e.g., bidirectional controlled state teleportation, controlled quantum secure direct communication, controlled quantum dialogue, etc.) have been studied using n -qubit ($n \geq 3$) entanglement. Specially, a large number of schemes for bidirectional controlled state teleportation are proposed using m -qubit entanglement ($m \in \{5, 6, 7\}$). Here, we propose a set of protocols to illustrate that it is possible to realize all these tasks related to controlled quantum communication using only Bell states and permutation of particles (PoP). As the generation and maintenance of a Bell state is much easier than a multi-partite entanglement, the proposed strategy has a clear advantage over the existing proposals. Further, it is shown that all the schemes proposed here may be viewed as applications of the concept of quantum cryptographic switch which was recently introduced by some of us. The performances of the proposed protocols as subjected to the amplitude damping and phase damping noise on the channels are also discussed.

1 Introduction

In 1993, Bennett *et al.* [1] introduced the fascinating idea of quantum teleportation. Since then a large number of modified teleportation schemes have been proposed. For example, all the existing schemes of quantum information splitting (QIS) or controlled teleportation (CT) [2, 3], quantum secret sharing (QSS) [4], hierarchical quantum information splitting (HQIS) [5, 6], remote state preparation [7], etc., can be viewed as modified teleportation schemes. The absence of any classical analogue and the potential applications in secure quantum communication and remote quantum operations [8] made the studies on teleportation and modified teleportation schemes extremely important. Bennett *et al.*'s original scheme [1] of quantum teleportation was a one-way scheme which enables Alice (sender) to transmit an unknown quantum state (single qubit quantum state) to Bob (receiver) by using two bits of classical communication and a pre-shared maximally entangled state. Later on, Huelga *et al.* [8, 9] and others generalized the original scheme of Bennett *et al.* to design the protocols for bidirectional state teleportation (BST). In a BST scheme, Alice and Bob are allowed to simultaneously transmit unknown quantum states to each other. Huelga *et al.* also established that BST can be used to implement nonlocal quantum gates. This can be visualized clearly, if we consider that Bob teleports a quantum state $|\psi\rangle$ to Alice, who applies a unitary operator U on $|\psi\rangle$ and teleports back the state $|\psi'\rangle = U|\psi\rangle$ to Bob. Thus, the existence of a scheme for BST essentially implies the ability to implement a nonlocal quantum gate or a quantum remote control. This is why BST schemes are very important for both quantum computation and quantum communication. Further extending the idea of BST, a good number of protocols for bidirectional controlled state teleportation (BCST) have been proposed in the recent past [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. A BCST scheme is a three party scheme, where BST is possible provided the supervisor/controller (Charlie) permits the other two users (Alice and Bob) to execute a protocol of BST. A careful study of all the recently proposed schemes of BCST [10, 11, 12, 14, 15, 16, 17, 18, 19] reveals that different n -qubit (with $n \geq 5$) entangled states are used in these protocols. Thus, at least 5 qubits are used in all the existing protocols of BCST. Keeping this in mind, in an earlier work [13], we explored the intrinsic symmetry of the 5-qubit quantum states that were used to propose the protocols of BCST until then. In Ref. [13], we reported a general structure of the quantum states that can be used for BCST as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle_{A_1 B_1} |\psi_2\rangle_{A_2 B_2} |a\rangle_{C_1} \pm |\psi_3\rangle_{A_1 B_1} |\psi_4\rangle_{A_2 B_2} |b\rangle_{C_1}), \quad (1)$$

where single qubit states $|a\rangle$ and $|b\rangle$ satisfy $\langle a|b\rangle = \delta_{a,b}$, $|\psi_i\rangle \in \{|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle, |\phi^-\rangle : |\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle\}$, $|\psi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$, $|\phi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$, and the subscripts A , B and C indicate the qubits of Alice, Bob and Charlie, respectively. Charlie prepares the state $|\psi\rangle$ and sends 1st and 3rd (2nd and 4th) qubits to Alice (Bob) and keeps the last qubit with himself. The condition

$$|\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle \quad (2)$$

	Initial state shared by Alice and Bob			
	$ \psi^+\rangle$	$ \psi^-\rangle$	$ \phi^+\rangle$	$ \phi^-\rangle$
SMO	Receiver	Receiver	Receiver	Receiver
00	I	Z	X	iY
01	X	iY	I	Z
10	Z	I	iY	X
11	iY	X	Z	I

Table 1: Perfect Teleportation. Here SMO stands for sender’s measurement outcome.

ensures that Charlie’s qubit is appropriately entangled with the remaining 4 qubits, and this in turn ensures that Alice and Bob are unaware of the entangled (Bell) states they share until Charlie measures his qubit using $\{|a\rangle, |b\rangle\}$ basis and discloses the outcome. Once Charlie discloses the measurement outcome, Alice and Bob know with certainty which two Bell states they share, and consequently they can use the conventional teleportation scheme to teleport unknown quantum states to each other by using classical communication and the unitary operations described in Table 1. It is important to note that in a successful protocol of BCST, Charlie should have control over both the directions of communication (i.e., both Alice to Bob and Bob to Alice transmission channels). We specifically note this because there exist a few proposals for BCST, where Charlie’s control is limited to one direction only [12, 14]. For example, the five qubit state used in Ref. [14] does not satisfy (2), and consequently it does not provide a scheme for controlled bidirectional teleportation (cf. Eq. (3) of Ref. [14] where the qubits 13 are separable from the qubits 245 implying that Alice and Bob can ignore Charlie’s control in one of the directions and teleport using the qubits 13). Similar shortcoming exists in the BCST protocol proposed by Li *et al.* [12], and the same was noted in our earlier work [13]. These two protocols do not qualify as the protocols of BCST. However, the other recently proposed protocols do not have this limitation.

In the above scheme, we require at least 5-qubit entanglement, if we choose $|a\rangle$ and $|b\rangle$ as single qubit states. One can of course achieve the same thing using a 6 or more qubit states by choosing $|a\rangle$ and $|b\rangle$ as multi-qubit states. For example, in Refs. [15, 16, 17, 18] BCST is reported using 6-qubit entangled states, and in Ref. [19] BCST is reported using a 7-qubit entangled state. In Ref. [15], it is explicitly shown that the 6-qubit entangled state used in [15] is of the form (1) where $|a\rangle$ and $|b\rangle$ were chosen as 2-qubit states and the six-qubit entangled state used in [18] can also be expressed in a form which can be viewed as an extended form of (1) that follows the same logic¹ (cf. Eqs. (11)-(12) of Ref. [15]). It is well-known that the experimental generation and maintenance of a 6-qubit or 7-qubit state is extremely difficult at the moment. This fact motivated us to look into the possibility of achieving BCST solely using Bell states (minimal quantum resource). Further, a few protocols for other controlled quantum communication tasks have been proposed in the recent past using multi-qubit states [14, 20, 21, 22]. Specifically, in Refs. [20, 21] 3-qubit GHZ states were used to design protocols for controlled quantum dialogue (CQD), and in Ref. [14] a 5-qubit quantum state was used to design a protocol of controlled quantum secure direct communication (CQSDC). In what follows, we will establish that it is possible to design protocols of CQD and CQSDC solely using Bell states.

In our earlier work on BCST [13], we just mentioned that BCST may be viewed as a quantum cryptographic switch [23], a concept introduced by us in a different context. Here, we elaborate the link and show that using the idea of quantum cryptographic switch, it is possible to construct schemes for BCST and other controlled quantum communication tasks solely using Bell states. Thus, to implement BCST, one does not require multi-partite entangled states as described in the recent papers [10, 11, 12, 13, 14, 15, 16, 17, 18, 19], rather one can construct protocols of BCST solely using Bell states and a technique known as permutation of particles (PoP), which was first introduced in 2003 by Deng and Long [24]. Since the pioneering work of Deng and Long, this interesting technique has been used in many protocols ([25, 26, 27, 28] and references therein). Before we describe our Bell-state-based protocols, it would be apt to briefly introduce the concept of quantum cryptographic switch, as it plays a crucial role in the construction of the new protocols described in the present paper. In a quantum scenario, a cryptographic switch describes a situation in which a controller/supervisor (Charlie) can control to a continuously varying degree the amount of information received by a semi-honest receiver (Bob), after a semi-honest sender (Alice) sends her information through a quantum channel. To understand the function of quantum cryptographic switch, let us consider that Charlie transmits a Bell state to Alice and Bob, but does not disclose which Bell state it is. Subsequently, Alice uses dense coding to transmit two bits of classical information to Bob. However, Bob can perfectly disclose the information encoded by Alice iff Charlie discloses the 2-bit of information corresponding to his choice of the Bell state. Charlie can continually vary the amount of disclosed information, and thus he can continuously alter the amount of information recovered by Bob. Further, we would like to note that in all the BCST protocols described until now, Alice and Bob need to be semi-honest as otherwise, they can completely ignore the supervisor Charlie and prepare and share entanglement by themselves. Such possibilities can be circumvented by specifically mentioning Alice and Bob as semi-honest (a user who follows the protocol but tries to cheat the controller). Unfortunately, this requirement was not specified in the existing protocols.

¹By the same logic we mean that unless Charlie measures his qubits and discloses the results Alice and Bob do not know which Bell states they share.

The remaining part of the present paper is organized as follows. In Section 2, we describe a Bell-state-based protocol of BCST. In Section 3, we describe a set of Bell-state-based protocols for controlled quantum communication. Specifically, we explicitly describe a protocol of CQD and discuss how to obtain similar protocols for CQSDC, controlled quantum key agreement (CQKA), controlled quantum key distribution (CQKD), etc. In Section 4, we study the effect of amplitude damping and phase damping noise on the Bell-state-based protocol for BCST. Finally, the paper is concluded in Section 5.

2 Bidirectional controlled teleportation using Bell states

If Alice and Bob are semi-honest then our protocol of BCST using Bell states works as follows

1. Charlie prepares $2n$ Bell states with $n \geq 1$ (each of them is randomly prepared in one of the Bell states²). He uses the Bell states to prepare 4 ordered sequences as follows:

- (a) A sequence with all the first qubits of the first n Bell states: $P_{A_1} = [p_1(t_A), p_2(t_A), \dots, p_n(t_A)]$,
- (b) A sequence with all the first qubits of the last n Bell states: $P_{A_2} = [p_{n+1}(t_A), p_{n+2}(t_A), \dots, p_{2n}(t_A)]$,
- (c) A sequence with all the second qubits of the first n Bell states: $P_{B_1} = [p_1(t_B), p_2(t_B), \dots, p_n(t_B)]$,
- (d) A sequence with all the second qubits of the last n Bell states: $P_{B_2} = [p_{n+1}(t_B), p_{n+2}(t_B), \dots, p_{2n}(t_B)]$.

where the subscript $1, 2, \dots, 2n$ denote the order of a particle pair $p_i = \{t_A^i, t_B^i\}$, which is in the Bell state.

2. Charlie randomizes the sequences of the second qubits, i.e., he applies n -qubit permutation operators Π_{n_1} and Π_{n_2} on P_{B_1} and P_{B_2} to create two new sequences as $P'_{B_i} = \Pi_{n_i} P_{B_i}$ with $i \in \{1, 2\}$ and sends these sequences to Bob. The actual order is known to Charlie only. Charlie also sends P_{A_1} and P_{A_2} to Alice. It is predeclared that the first (last) n Bell states prepared by Charlie are to be used for Alice to Bob (Bob to Alice) teleportation.
3. After receiving the qubits from Charlie, Alice (Bob) understands that she (he) can now teleport her (his) unknown qubits $|\psi_{A_j}\rangle = \alpha_{A_j}|0\rangle + \beta_{A_j}|1\rangle : |\alpha_{A_j}|^2 + |\beta_{A_j}|^2 = 1$ ($|\psi_{B_j}\rangle = \alpha_{B_j}|0\rangle + \beta_{B_j}|1\rangle : |\alpha_{B_j}|^2 + |\beta_{B_j}|^2 = 1$) to Bob (Alice) using standard teleportation scheme, i.e., by entangling her (his) unknown qubit $|\psi_{A_j}\rangle$ ($|\psi_{B_j}\rangle$) with $p'_j(t_A)$ ($p'_{n+j}(t_B)$) and subsequently measuring both the qubits in computational basis and communicating the result to Bob (Alice). Since the sequence with Alice and Bob are different, even if Alice or Bob obtain the access of both P_{A_i} and P'_{B_i} , they will not be able to find out the Bell states prepared by Charlie. Thus, any kind of collusion between Alice and Bob would not help Alice and Bob to circumvent the control of Charlie as long as they are semi-honest. Specifically, even after Alice (Bob) communicates the outcome of her (his) measurement in computational basis to Bob (Alice), he (she) will not be able to reproduce the unknown state sent by Alice (Bob) as he (she) neither knows which Bell state was prepared by Charlie nor knows which qubit was entangled with which qubit. Interestingly, the existing protocols of BCST are not protected under such a collusion between Alice and Bob as in all the protocols that uses quantum states of the form (1) Alice (Bob) can send her qubits to Bob (Alice) so that Bob (Alice) can perform an appropriate Bell measurement to know the nature of Bell state and return the particles to Alice (Bob).
4. When Charlie plans to allow Bob (Alice) to reconstruct the unknown quantum state teleported by Alice (Bob), then Charlie discloses the Bell state which he had prepared and the exact sequence Π_{n_1} (Π_{n_2}).
5. Since the initial Bell states and the exact sequence are known, Bob (Alice) now applies appropriate unitary operations on his (her) qubits as described in Table 1 and obtains the unknown quantum states sent by Alice (Bob).

This protocol has certain advantages over the existing protocols of BCST. Firstly, it requires only 2-qubit entanglement which is much easier to produce and maintain. Secondly, the control of Charlie on Alice and Bob is much more compared to that in the existing protocols. To be precise, using Π_{n_1} and Π_{n_2} Charlie can separately control Alice to Bob and Bob to Alice teleportation channels. For example, if he discloses Π_{n_1} and the nature of first n Bell states prepared by him but keeps Π_{n_2} and the nature of the last n Bell states secret, then Bob will be able to reproduce the state teleported by Alice, but Alice will not be able to obtain the state teleported by Bob. Such directional control was missing in earlier protocols of BCST. For example, in all the protocols that use a state of the form (1), when Charlie announces his measurement outcome in $\{|a\rangle, |b\rangle\}$ basis, then both Alice and Bob will know the entangled states shared by them and will be able to reconstruct the state teleported by their partners. Further, in all the previous protocols Alice (Bob) always prepares the state sent by Bob (Alice) with perfect fidelity if the channel is perfect. Charlie, has no control over the fidelity with which Bob (Alice) can reconstruct the state teleported by Alice (Bob). However, in the present protocol, which is a variant of

²We can assume that Charlie has a quantum random number generator, and he has generated a large sequence of 0 and 1 through it. He uses the outcomes of the random number generator to decide which Bell state is to be prepared. For example, we may consider that if the first two bit values obtained from the random number generator are 00, 01, 10, and 11 then he prepares $|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle$, and $|\phi^-\rangle$, respectively. Here, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are Bell states.

quantum cryptographic switch, maximum fidelity of the state prepared by Alice and Bob can be controlled by Charlie. This is so because in the present protocol Charlie has the freedom to decide whether he will precisely disclose which Bell state he had prepared (a 2-bit information) in a particular position. For example, Charlie may choose to inform Bob that the first Bell state prepared by him was such that the parity-0 Bell states are twice more likely than the parity-1 states and that Bell states of equal parity are equally likely. This corresponds to a probability distribution of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, i.e., an entropy of about 1.92 bits, implying that Charlie reveals $c = 0.08$ bits. Consequence of such an incomplete disclosure is that Bob will not be able to uniquely decide which unitary operations to be applied, and consequently the fidelity of the state reconstructed by Bob will reduce. In brief, the BCST protocol proposed here requires lesser quantum resources and provides better control. Further, it may be modified to a multi-controlled BST (MCBST) scheme where first n Bell states are prepared by Charlie 1 and the last n Bell states are prepared by Charlie 2. Thus, the BST scheme will not work unless both of them disclose the exact sequences and what Bell states were prepared by them. However, this simple idea of MCBST has a limitation that one Charlie controls one channel (say Alice to Bob) and the other Charlie controls the other Channel (say Bob to Alice). A more sophisticated idea of MCBST will be separately discussed in a future work.

3 Other protocols of controlled quantum communication using Bell states

A large number of entangled-state-based quantum communication protocols for different communication tasks have been proposed in the recent past. For example, protocols for quantum dialogue (QD) [29, 30], quantum key distribution (QKD) [31, 32], quantum secure direct communication (QSDC), deterministic secure quantum communication (DSQC) [26, 27, 28, 33, 34, 35] have been proposed. Using the PoP-based trick described above, one can obtain the controlled versions of all these two-party (mostly bidirectional) quantum communication schemes just by using Bell states. In what follows, we briefly describe how to modify the existing two-party protocols of quantum communication to obtain the corresponding controlled three-party protocols of quantum communication. To begin with, we describe a CQD protocol of Ba An type in the following subsection and subsequently describe how to transform the proposed protocol of CQD into a Ping-Pong (PP)-type protocol of CQSDC and a protocol of CQKA.

3.1 Controlled quantum dialogue protocol of Ba An type

Let us first describe the Ba An's original two-party scheme of QD in which both Alice and Bob can simultaneously communicate. This simple scheme can be described in the following steps [29]:

- Step 1** Bob prepares $|\phi^+\rangle^{\otimes n} : |\phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ and keeps the first qubit of each Bell state ($|\phi^+\rangle$) with himself as home qubit and encodes his secret message by using the following rule: He encodes 00, 01, 10 and 11 by applying unitary operations I , X , iY and Z , respectively on the second qubit.
- Step 2** Bob sends a sequence of the second qubits (travel qubits) to Alice and confirms that Alice has received the qubits.
- Step 3** Alice encodes her secret message by using the same rule and returns the travel qubits to Bob. After receiving the encoded travel qubits Bob appropriately combines them with the sequence of home qubits and performs Bell measurements on the partner particles that were initially prepared in $|\phi^+\rangle$.
- Step 4** Alice discloses whether it was a run in message mode (MM) or in control mode (CM). In MM, Bob decodes Alice's bits and announces the outcome of Bell measurement performed by him. Alice uses that result to decode the bits encoded by Bob. In CM, Alice announces her encoding value and Bob uses that for eavesdropping check.

It is straightforward to convert this two-party protocol into an equivalent three-party protocol, where Charlie is the controller and Alice and Bob are semi-honest users who want to execute a scheme of QD. The modified protocol with clear measures of security works as follows:

- Step 1** Charlie prepares n copies of a Bell state $|\phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$. He prepares two sequences: the first sequence P_{B_1} is prepared with all the first qubits of Bell pairs and the second sequence P_{B_2} is prepared with all the second qubits.
- Step 2** Charlie applies n -qubit permutation operators Π_n on P_{B_1} to create a new sequence $P'_{B_1} = \Pi_n P_{B_1}$ and sends both P'_{B_1} and P_{B_2} to Bob.
- Step 3** Bob uses the qubits of P'_{B_1} (P_{B_2}) as home (travel) qubits. He encodes his secret message 00, 01, 10 and 11 by applying unitary operations U_0, U_1, U_2 and U_3 respectively on the second qubit (i.e., on the qubits of sequence P_{B_2}). Without loss of generality, we may assume that $U_0 = I$, $U_1 = \sigma_x = X$, $U_2 = i\sigma_y = iY$ and $U_3 = \sigma_z = Z$, where σ_i are Pauli matrices. Further, we assume that after the encoding operation the sequence P_{B_2} transforms to Q_{B_2} .
- Step 4** Bob first prepares n decoy qubits in a random sequence of $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$, i.e., the decoy qubit state is $\otimes_{j=1}^n |P_j\rangle$, $|P_j\rangle \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$. Bob then randomly inserts the decoy qubits in Q_{B_2} to obtain an enlarged new sequence R_{B_2} and sends that to Alice and confirms that Alice has received the entire sequence.

Step 5 Bob discloses the positions of decoy qubits, and applies BB84 subroutine³ in collaboration with Alice and thus computes the error rate. If the error rate exceeds the tolerable limit, then Alice and Bob abort this communication and repeat the procedure from the beginning. Otherwise, they go on to the next step.

All the intercept-resend attacks are detected in this step. Any attack by Eve will not provide her any meaningful information about the encoding operation executed by Bob as Eve's access to the Bell state is limited to a single qubit.

Step 6 Alice encodes her secret message by using the same set of encoding operations as was used by Bob and subsequently randomly inserts a set of n decoy qubits in her sequence and returns the new sequence R_{B_3} obtained by this method to Bob.

Step 7 After Bob confirms that he has received R_{B_3} , Alice discloses the positions of the decoy qubits, and Alice and Bob follow Step 5 to check eavesdropping. If no eavesdropping is found they move to the next step.

Step 8 Charlie announces the exact sequence of P_{B_1} .

Step 9 Bob uses the information obtained from Charlie to create n Bell pairs and performs Bell measurements on them. Subsequently, he announces the outcomes of his Bell measurements. As Bob knows the initial Bell state, final Bell state and his own encoding operation he can decode Alice's bits. Similarly, Alice uses the results of Bell measurements announced by Bob, knowledge of the initial state and her own encoding operation to decode Bob's bits.

Note that unless Charlie discloses the exact sequence, Alice and Bob, who are assumed to be semi-honest, cannot decode each other's information. Thus, we have a CQD protocol. Now, it is easy to visualize that we can restrict the protocol to the extent that only Alice communicates a message (i.e., Bob neither encodes anything nor announces the outcomes of his Bell measurements) then the above protocol will reduce to a PP-type protocol of CQSDC in which Alice can communicate a secure message to Bob. In the original PP protocol [33], only 2 encoding operations were used, and subsequently its efficiency was increased in Cai Li (CL) protocol [34] by including all 4 encoding operations that are used here. Thus, the above protocol can actually provide us an efficient controlled PP protocol, or equivalently a controlled CL protocol. Further, we know that all QSDC protocols can be reduced to QKD protocols as Alice may choose to communicate a random sequence instead of a meaningful message. In that case, the QSDC protocol obtained by modifying the CQD protocol will reduce to a protocol of CQKD. Interestingly, the above protocol can also be transformed to a protocol of CQKA. Where both Alice and Bob contributes equally to the final key. It is easy to visualize if we consider that Alice and Bob possess two random strings of keys (say k_A and k_B) and they use the CQD protocol described above to communicate these keys to each other and decide that for all future communications they will use $k = k_A \oplus k_B$ as the shared key. This relation between QKA and QD is elaborately discussed in our earlier work [37]. Before we finish this section, we would like to note that in the protocol proposed here, we have used the BB84 subroutine for eavesdropping check, where decoy qubits were prepared and measured randomly in X -basis and Z -basis. One can replace this by a GV-type subroutine described in [25], where decoy qubits are prepared as Bell states and thus can perform all the tasks described above (i.e., CQD, CQSDC, CQKA, CQKD, etc.) using orthogonal states alone. This is interesting as not many orthogonal state based protocols of quantum communication are proposed until now.

We have already described a few protocols of controlled quantum communication solely using Bell states and PoP technique. The idea is novel as so far usually n -partite ($n \geq 3$) entanglement is used to achieve controlled quantum communication protocols. Further, in the present paper we have restricted ourselves to a group of protocols of quantum communication and showed that they may be modified to obtain corresponding controlled protocols. Same strategy may be used to generalize other entangled-state-based protocols of quantum communication, too. For example, CL protocol [34], DLL protocol [35], etc., can be easily modified to obtain their controlled versions.

4 Effect of noise

In this section, we will investigate the effect of noise on the BCST schemes and the other schemes proposed by us. To begin with, let us note a few important things: (i) As the qubits to be teleported are never available in the channel, we can assume that the channel noise does not have any effect on them. Thus, it would be sufficient to study the effect of noise on the qubits that travel through the channel (travel qubits). (ii) Following the same logic, we can assume that Charlie's qubit is independent of noise.

As Charlie's measurement reduces the 5-qubit quantum state used earlier by us in Ref. [13] to a product of two Bell states, and only these qubits are exposed to noise, we can easily understand that the effect of noise on the 5-qubit state based BCST would be the same as that in the Bell-state-based BCST protocol proposed here. Keeping this in mind, here

³BB84 subroutine [36] means eavesdropping is checked by following a procedure similar to that adopted in the original BB84 protocol. Specifically, BB84 subroutine implies that Alice (Bob) randomly selects half of the qubits received by her (him) to form a verification string. She (He) measures the verification qubits randomly in $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ basis and announces the measurement outcome, the position of that qubit in the string and the basis used for the particular measurement. Bob (Alice) also measures the corresponding qubit using the same basis (if needed) and compares his (her) result with the announced result of Alice (Bob) to detect eavesdropping.

we study the effects of different kinds of noise channels on the Bell-state-based scheme for BCST. The results obtained here are also applicable to the 5-qubit state based BCST scheme under the above assumptions.

It is well known that the amplitude-damping noise model is described by the following set of Kraus operators [38]:

$$E_0^A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta_A} \end{bmatrix}, \quad E_1^A = \begin{bmatrix} 0 & \sqrt{\eta_A} \\ 0 & 0 \end{bmatrix}, \quad (3)$$

where the decoherence rate η_A ($0 \leq \eta_A \leq 1$) describes the probability of error due to amplitude-damping noisy environment when a travel qubit passes through it. Similarly, the following set of Kraus operators describes the phase-damping channel [38]:

$$E_0^P = \sqrt{1-\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1^P = \sqrt{\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2^P = \sqrt{\eta_P} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4)$$

where η_P ($0 \leq \eta_P \leq 1$) is the decoherence rate for the phase-damping noise. In what follows, we assume that Alice (Sender1) and Bob (Sender2) wish to teleport qubits $|\zeta_1\rangle_{S'_1} \equiv a_1|0\rangle + b_1 \exp(i\phi_1)|1\rangle$, and $|\zeta_2\rangle_{S'_2} \equiv a_2|0\rangle + b_2 \exp(i\phi_2)|1\rangle$ to Bob (Receiver1) and Alice (Receiver2), respectively through a noisy channel. The channel is either phase damping or amplitude damping channel. These assumptions help us to study the effect of these two types of noise channels independently.

4.1 Effect of noise on the protocols of CBST

Consider that Charlie has distributed two Bell states to Alice and Bob as $|\psi\rangle_{S_1 R_1 S_2 R_2} = |\psi_1\rangle_{S_1 R_1} |\psi_2\rangle_{S_2 R_2}$ with $|\psi_i\rangle \in \{|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle, |\phi^-\rangle\}$. Now, Sender1 and Sender2 wish to teleport qubits $|\zeta_1\rangle_{S'_1}$ and $|\zeta_2\rangle_{S'_2}$ to Receiver1 and Receiver2, respectively. Thus, the combined state of the system is $|\psi'\rangle_{S_1 R_1 S_2 R_2 S'_1 S'_2} = |\psi_1\rangle_{S_1 R_1} \otimes |\psi_2\rangle_{S_2 R_2} \otimes |\zeta_1\rangle_{S'_1} \otimes |\zeta_2\rangle_{S'_2}$ and the corresponding density matrix (after suitable rearrangement of the particles) is

$$\rho = |\psi\rangle_{S_1 S'_1 R_1 S_2 S'_2 R_2 S_1 S'_1 R_1 S_2 S'_2 R_2} \langle \psi|.$$

Now, the noisy environment described by (3) or (4) transforms the density operator ρ as

$$\rho_k = \sum_{i,j} E_{i,S_1}^k \otimes I_{2,S'_1} \otimes E_{j,R_1}^k \otimes E_{j,S_2}^k \otimes I_{2,S'_2} \otimes E_{i,R_2}^k \rho (E_{i,S_1}^k \otimes I_{2,S'_1} \otimes E_{j,R_1}^k \otimes E_{j,S_2}^k \otimes I_{2,S'_2} \otimes E_{i,R_2}^k)^\dagger. \quad (5)$$

For simplicity, we assume that both the qubits sent to Alice (i.e., S_1 and R_2 qubits) are affected by the same Kraus operator (same noise) and similarly, the qubits R_1 and S_2 sent to Bob are also affected by the same Kraus operator. This assumption is justified as Charlie to Alice communication is done by a quantum channel and Charlie to Bob communication is done by another quantum channel. In accordance with the Bell-state based CBST scheme described above, S_1 and S'_1 qubits are measured by Alice using computational basis, whereas S_2 and S'_2 qubits are measured by Bob using the same basis. Here, we also assume that the measurements of both Alice and Bob yield $|00\rangle$. To selectively choose these outcomes, we have to apply the unitary operator

$$U = (|00\rangle_{S_1 S'_1 S_1 S'_1} \langle 00|) \otimes I_{2,R_1} \otimes (|00\rangle_{S_2 S'_2 S_2 S'_2} \langle 00|) \otimes I_{2,R_2}$$

on ρ_k and that would yield an unnormalized quantum state

$$\rho_{k_1} = U \rho_k U^\dagger,$$

which can be normalized to

$$\rho_{k_2} = \frac{\rho_{k_1}}{\text{Tr}(\rho_{k_1})}.$$

Now, the combined state of the qubits of Receivers 1 and 2 (i.e., the state of the qubits R_1 and R_2) in a noisy environment or ρ_{k_3} can be obtained from ρ_{k_2} by tracing out the qubits that are already measured. Specifically,

$$\rho_{k_3} = \text{Tr}_{S_1 S'_1 S_2 S'_2} (\rho_{k_2}).$$

Depending upon the initial Bell states and the measurement outcomes of the senders, the receiver(s) may have to apply appropriate Pauli operators on ρ_{k_3} to obtain the final quantum state $\rho_{k,\text{out}}$, where the index k denotes a specific noise model. In the present case, we have already assumed that the outcomes of measurements of both the senders are $|00\rangle$, if we further assume that the initial state prepared by Charlie as $|\psi^+\rangle^{\otimes 2}$, the receivers would not require to apply any unitary operator (in other words, the receivers need to apply the identity operators only). We have already assumed that Alice (Sender1) and Bob (Sender2) wish to teleport qubits $|\zeta_1\rangle_{S'_1} = a_1|0\rangle + b_1 \exp(i\phi_1)|1\rangle$ and $|\zeta_2\rangle_{S'_2} = a_2|0\rangle + b_2 \exp(i\phi_2)|1\rangle$ at the side of Bob (Receiver1) and Alice (Receiver2), respectively. Thus, in the absence of noise, the expected final state is a product state

$$|T\rangle_{R_1 R_2} = (a_1|0\rangle + b_1 \exp(i\phi_1)|1\rangle) \otimes (a_2|0\rangle + b_2 \exp(i\phi_2)|1\rangle),$$

where Alice (Receiver2) will have qubit $a_2|0\rangle + b_2 \exp(i\phi_2)|1\rangle$ in her possession, and Bob (Receiver1) will have $a_1|0\rangle + b_1 \exp(i\phi_1)|1\rangle$. As a_i and b_i are real, and $a_i^2 + b_i^2 = 1$, we can assume that $a_i = \sin\theta_i$ and $b_i = \cos\theta_i$ with $i \in \{1, 2\}$. Thus,

$$|T\rangle_{R_1 R_2} = \sin\theta_1 \sin\theta_2 |00\rangle + \cos\theta_2 \sin\theta_1 \exp(i\phi_2) |01\rangle + \cos\theta_1 \sin\theta_2 \exp(i\phi_1) |10\rangle + \cos\theta_1 \cos\theta_2 \exp(i(\phi_1 + \phi_2)) |11\rangle.$$

The effect of noise can now be investigated by comparing the quantum state $\rho_{k,\text{out}}$ produced in the noisy environment with the state $|T\rangle_{R_1 R_2}$ using fidelity

$$F = \langle T | \rho_{k,\text{out}} | T \rangle, \quad (6)$$

which is the square of the conventional definition of fidelity⁴.

As we have assumed that our Bell-state-based protocol for CBST starts with the initial state

$$|\psi\rangle = |\psi^+\rangle_{S_1 R_1} |\psi^+\rangle_{S_2 R_2}, \quad (7)$$

and the outcomes of the measurements of both the receivers are $|00\rangle$, the above described method of obtaining $\rho_{k,\text{out}}$ yields

$$\rho_{A,\text{out}} = N_A \begin{pmatrix} \frac{(1+\eta_A)^4 \cos^2\theta_1 \cos^2\theta_2}{(1-\eta_A)^4} & \frac{\cos^2\theta_1 \sin 2\theta_2 \exp(-i\phi_2)}{2(1-\eta_A)^3} & \frac{\sin 2\theta_1 \cos^2\theta_2 \exp(-i\phi_1)}{2(1-\eta_A)^3} & \frac{\sin 2\theta_1 \sin 2\theta_2 \exp(-i\phi_{12})}{4(1-\eta_A)^2} \\ \frac{\cos^2\theta_1 \sin 2\theta_2 \exp(i\phi_2)}{2(1-\eta_A)^3} & \frac{\{\cos^2\theta_1 \sin^2\theta_2 + \eta_A^2 \sin^2\theta_1 \cos^2\theta_2\}}{(1-\eta_A)^2} & \frac{\sin 2\theta_1 \sin 2\theta_2 \exp(-i\Delta\phi)}{4(1-\eta_A)^2} & \frac{\sin 2\theta_1 \sin^2\theta_2 \exp(-i\phi_1)}{2(1-\eta_A)} \\ \frac{\sin 2\theta_1 \cos^2\theta_2 \exp(i\phi_1)}{2(1-\eta_A)^3} & \frac{\sin 2\theta_1 \sin 2\theta_2 \exp(i\Delta\phi)}{4(1-\eta_A)^2} & \frac{\{\eta_A^2 \cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2\}}{(1-\eta_A)^2} & \frac{\sin^2\theta_1 \sin 2\theta_2 \exp(-i\phi_2)}{2(1-\eta_A)} \\ \frac{\sin 2\theta_1 \sin 2\theta_2 \exp(i\phi_{12})}{4(1-\eta_A)^2} & \frac{\sin 2\theta_1 \sin^2\theta_2 \exp(i\phi_1)}{2(1-\eta_A)} & \frac{\sin^2\theta_1 \sin 2\theta_2 \exp(i\phi_2)}{2(1-\eta_A)} & \sin^2\theta_1 \sin^2\theta_2 \end{pmatrix}, \quad (8)$$

and

$$\rho_{P,\text{out}} = N_P \begin{pmatrix} P_{11} \cos^2\theta_1 \cos^2\theta_2 & 2 \cos^2\theta_1 \sin 2\theta_2 \exp(-i\phi_2) & 2 \sin 2\theta_1 \cos^2\theta_2 \exp(-i\phi_1) & \sin 2\theta_1 \sin 2\theta_2 \exp(-i\phi_{12}) \\ 2 \cos^2\theta_1 \sin 2\theta_2 \exp(i\phi_2) & 4 \cos^2\theta_1 \sin^2\theta_2 & \sin 2\theta_1 \sin 2\theta_2 \exp(-i\Delta\phi) & 2 \sin 2\theta_1 \sin^2\theta_2 \exp(-i\phi_1) \\ 2 \sin 2\theta_1 \cos^2\theta_2 \exp(i\phi_1) & \sin 2\theta_1 \sin 2\theta_2 \exp(i\Delta\phi) & 4 \sin^2\theta_1 \cos^2\theta_2 & 2 \sin^2\theta_1 \sin 2\theta_2 \exp(-i\phi_2) \\ \sin 2\theta_1 \sin 2\theta_2 \exp(i\phi_{12}) & 2 \sin 2\theta_1 \sin^2\theta_2 \exp(i\phi_1) & 2 \sin^2\theta_1 \sin 2\theta_2 \exp(i\phi_2) & P_{11} \sin^2\theta_1 \sin^2\theta_2 \end{pmatrix}, \quad (9)$$

where $\phi_{12} = \phi_1 + \phi_2$, $\Delta\phi = (\phi_1 - \phi_2)$, $P_{11} = \frac{4(1-2\eta_P+2\eta_P^2)^2}{(1-\eta_P)^4}$,

$$N_A = \frac{4(1-\eta_A)^4}{2[(2-4\eta_A+5\eta_A^2-4\eta_A^3+2\eta_A^4)+\eta_A(2-3\eta_A+2\eta_A^2)\{\cos 2\theta_1+\cos 2\theta_2\}+\eta_A^2 \cos 2\theta_1 \cos 2\theta_2]},$$

and

$$N_P = \frac{(1-\eta_P)^4}{2[(2-8\eta_P+14\eta_P^2-12\eta_P^3+5\eta_P^4)+2\eta_A^2(2-4\eta_P+3\eta_P^2)\cos 2\theta_1 \cos 2\theta_2]}.$$

Using (6) and (8) we obtain the fidelity of the quantum state prepared using the proposed CBST scheme under amplitude-damping noise as

$$\begin{aligned} F_{\text{AD}} &= \frac{1}{16(2-4\eta_A+5\eta_A^2-4\eta_A^3+2\eta_A^4+\eta_A^2 \cos 2\theta_1 \cos 2\theta_2 + \eta_A(2-3\eta_A+2\eta_A^2)(\cos 2\theta_1 + \cos 2\theta_2))} \\ &\times [32 - 164\eta_A + 57\eta_A^2 - 26\eta_A^3 + 10\eta_A^4 + \eta_A(34 - 51\eta_A + 30\eta_A^2)(\cos 2\theta_1 + \cos 2\theta_2) \\ &+ \eta_A^2(3 - 2\eta_A + 2\eta_A^2)(\cos 4\theta_1 + \cos 4\theta_2) + 4\eta_A^3(3 - 2\eta_A + 2\eta_A^2)(\cos 2\theta_1 \cos 4\theta_2 + \cos 4\theta_1 \cos 2\theta_2) \\ &+ 16\eta_A^2(2 - 2\eta_A + \eta_A^2) \cos 2\theta_1 \cos 2\theta_2 + \eta_A^2(1 - 2\eta_A + 2\eta_A^2) \cos 4\theta_1 \cos 4\theta_2] \end{aligned} \quad (10)$$

Similarly, by using (6) and (9) we obtain the fidelity of the quantum state prepared using the proposed CBST scheme under phase-damping noise as

$$F_{\text{PD}} = \frac{32-128\eta_P+210\eta_P^2-164\eta_P^3+59\eta_P^4+\eta_P^2\{2-4\eta_P+3\eta_P^2\}(16\cos 2\theta_1 \cos 2\theta_2 + \cos 4\theta_1 \cos 4\theta_2 + 3(\cos 4\theta_1 + \cos 4\theta_2))}{16(2-8\eta_P+14\eta_P^2-12\eta_P^3+5\eta_P^4+\eta_P^2\{2-4\eta_P+3\eta_P^2\}\cos 2\theta_1 \cos 2\theta_2)}. \quad (11)$$

Following a similar strategy, we can also obtain analytic expressions for the fidelity for the CQD protocol described above in a noisy environment. The analytic expressions of fidelity would depend on the Charlie's choice of initial state and Alice's and Bob's secret messages (i.e., on unitary operators corresponding to their secret messages). Expressions for fidelity obtained for various situations are listed in Table 2.

⁴Usually fidelity $F(\sigma, \rho)$ of two quantum states ρ and σ is defined as $F(\sigma, \rho) = \text{Tr} \sqrt{\sigma^{\frac{1}{2}} \rho \sigma^{\frac{1}{2}}}$. However, in the present work, we have used (6) as the definition of fidelity.

Initial state	Operations of Alice	Operations of Bob	Fidelity in amplitude damping channel (F'_{AD})	Fidelity in phase damping channel (F'_{PD})
$ \psi^\pm\rangle$	X, iY	X, iY	$F'_{AD1} = \frac{4-8\eta_A+7\eta_A^2-2\eta_A^3+\eta_A^4}{4(1-\eta_A+\eta_A^2)}$	$F'_{PD1} = \frac{2-6\eta_A+8\eta_A^2-4\eta_A^3+\eta_A^4}{2(1-2\eta_P+2\eta_P^2)}$
	I, Z	I, Z	$F'_{AD2} = \frac{4-8\eta_A+9\eta_A^2-4\eta_A^3+\eta_A^4}{4(1-\eta_A+\eta_A^2)}$	
	X, iY	I, Z	$F'_{AD3} = \frac{(1-\eta_A)^2(4+\eta_A^2)}{4(1-\eta_A+\eta_A^2)}$	
	I, Z	X, iY	$F'_{AD4} = \frac{4-8\eta_A+7\eta_A^2-4\eta_A^3+\eta_A^4}{4(1-\eta_A+\eta_A^2)}$	
$ \phi^\pm\rangle$	I, X, iY, Z	I, X, iY, Z	$F'_{AD5} = \frac{1}{4}(2-\eta_A)^2$	$F'_{PD2} = \frac{1}{2}(2-2\eta_P+\eta_P^2)$

Table 2: Analytic expressions of fidelity for the proposed CQD protocol in amplitude and phase damping channels for all possible situations. Here the $'$ in superscript is used to distinguish the fidelities of CQD protocol from that of the BCST protocols. Here, we have considered that initially Charlie prepared a Bell state $|\psi\rangle \in \{|\psi^\pm\rangle, |\phi^\pm\rangle\}$ and sent both the qubits to Alice. As both the qubits went through the same channel, same Kraus operator worked on them. Later one of the qubits travelled from Alice to Bob and Bob to Alice in a noisy environment.

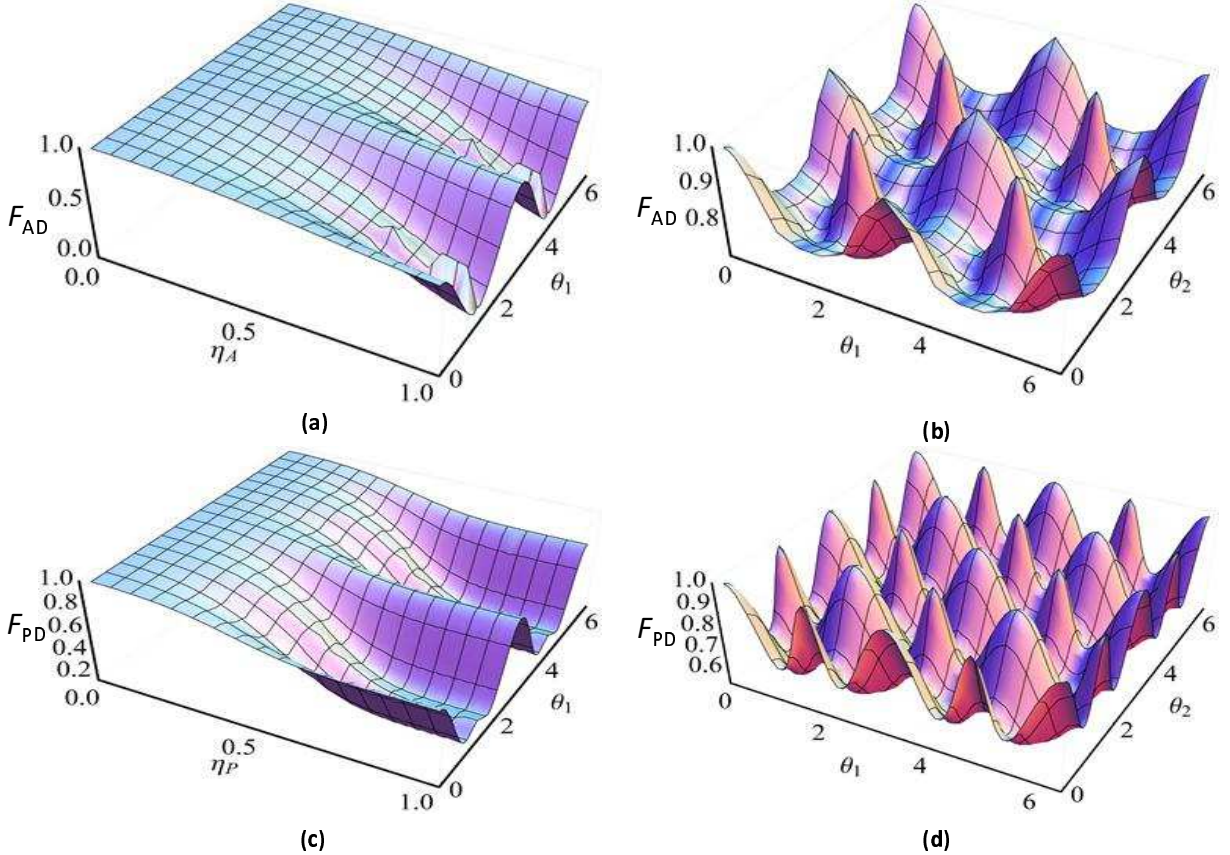


Figure 1: (Color online) Comparison of the effects of amplitude damping and phase damping noise on the fidelity in Bell-state-based BCST protocol. (a) and (b) show the effect of amplitude-damping noise, and (c) and (d) show the effect of phase-damping noise on the fidelity of the Bell-state-base BCST protocol. (a) and (c) show the effects of noise on the fidelity for $\theta_2 = \frac{\pi}{6}$. (b) and (d) show the effect of noise for $\eta = 0.5$.

From (10)-(11) we can observe that the fidelities F_{AD} and F_{PD} are independent of the corresponding phase information ϕ_i , and they depend only on the decoherence rate η_k and the amplitude information (i.e., a_i and b_i) of the states to be teleported. As the amplitude of the state used is fixed in the CQD protocol, fidelities reported in Table 2 are naturally function of η_k only. These observations are consistent with the recent observations reported in Ref. [39] in the context of joint remote state preparation (JRSP) in noisy environment and in Ref. [40] in context of controlled bidirectional RSP (CBRSRSP). The method adopted to study the effect of noise in the present paper and in Refs. [39, 40] is quite general and can be easily applied to other schemes of quantum communication and to study the effect of other noise models, such

as, generalized amplitude damping (GAD) channel or squeezed generalized amplitude damping (SGAD)⁵ channel [23, 41]. Specifically, to study the effect of SGAD or GAD channels on the fidelity, we will just require to replace the Kraus operators used above by the Kraus operators of SGAD or GAD channels, respectively. Similarly, one may study the effect of Pauli-type noise, too [42]. However, here we restrict ourselves to amplitude-damping and phase-damping channels only. In case the above described Bell-state-based BCST scheme is realized in a noisy environment, then the fidelity corresponding to various noise models would depend on the various parameters as shown in Figs. 1-2.

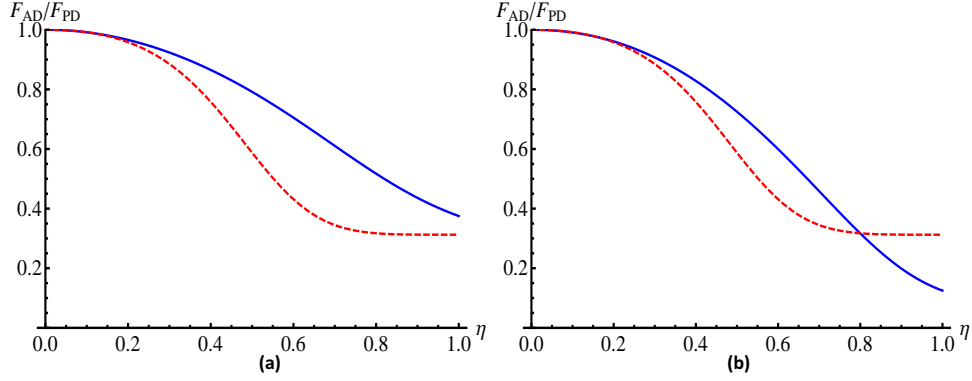


Figure 2: (Color online) Comparison of the effects of amplitude damping and phase damping noise on the fidelity of Bell-state-based BCST scheme. The smooth (blue) and dashed (red) lines correspond to the effects of amplitude-damping and phase-damping channels on the fidelity of the Bell-state-based scheme of BCST, respectively. The first plot shows the effect of noise on the fidelities for $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{6}$, and the second plot shows the effect of noise on the fidelities for $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{3}$.

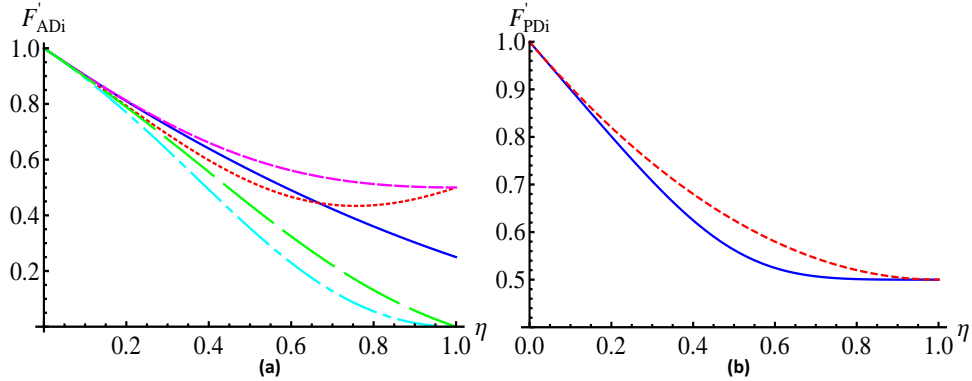


Figure 3: (Color online) Comparison of the effects of amplitude damping and phase damping noise on the fidelity for the protocol of CQD. (a) Compares all the fidelity expressions for amplitude damping channel ($F_{AD1} - F_{AD5}$) provided in the 4th column of Table 2. Smooth (blue) line corresponds to F'_{AD5} , i.e., initial state $|\phi^\pm\rangle$, and dotted (red), small dashed (magenta), dot-dashed (cyan) and large dashed (green) lines correspond to F'_{AD1} , F'_{AD2} , F'_{AD3} and F'_{AD4} , respectively (i.e., for various encoding operations of Alice and Bob for the initial state as $|\psi^\pm\rangle$). (b) The smooth (blue) and dashed (red) lines in the second plot correspond to the effects of phase-damping channel for the CQD protocol realized with the initial state $|\psi^\pm\rangle$ and $|\phi^\pm\rangle$, respectively. Clearly, $|\phi^\pm\rangle$ provides a better choice as it leads to higher fidelity.

Specifically, Fig. 2 compares the fidelity in amplitude damping and phase damping channels (solid and dashed lines, respectively) in CBST for two independent choices of unknown states to be teleported. In Fig. 2 a, we can easily observe that the amplitude damping channel gives higher fidelity than the phase damping channel for all values of decoherence rate $\eta_A = \eta_P = \eta$. However, for a different choice of states to be teleported, we obtain that even phase damping channel can have higher fidelity, especially at higher values of η (cf. Fig. 2 b). Thus, we may conclude that the amplitude damping channels do not have always fidelity greater than that in the phase damping channels for the same value of decoherence rate. The fact is further illustrated in Fig. 1, where the variation in the fidelity is discussed in two cases: (1) Considering that Bob teleports a state with fixed value of θ_2 (i.e., fixed value of a_2 and b_2), which represents a quantum state from a family of quantum states that differ only by the value of the relative phase ϕ_2 . As fidelity is independent of ϕ_2 , all such states are equivalent for our purpose. Specifically, variation in fidelity for this particular case is shown in Fig. 1 (2) Considering that $\eta_k = \eta = 0.5$ for both the directions of communication (i.e., for Alice to Bob and Bob to Alice communication). Here, it is

⁵AD and GAD channels are the special cases of SGAD channel.

important to note that from the symmetry in the expressions of the fidelity in both types of noise channels, we can observe that fidelity should remain unchanged if Alice and Bob interchange the unknown state they want to teleport to each other using the same quantum channel. Finally, the effects of amplitude and phase damping channels on the fidelity for CQD protocol are illustrated in Fig. 3. From Fig. 3 b, it can be easily seen that in presence of a phase damping channel, the fidelity for the CQD protocol that uses $|\phi^\pm\rangle$ as initial state is always greater than that which uses $|\psi^\pm\rangle$ as the initial state. Thus, for a phase damping channel, it would be beneficial to start with $|\phi^\pm\rangle$ as the initial state. However, no such preference is observed in case of amplitude damping channel (cf. Fig. 3 a). Further, from Table 2, we can observe that the fidelities in the presence of phase damping noise are only dependent on the initial state prepared by Charlie, while for the amplitude damping channel, the fidelities are dependent on both the initial state prepared by Charlie and the operations of Alice and Bob.

5 Conclusions

We have already mentioned that in the recent past, several schemes for controlled quantum communication (e.g., BCST, CQSDC, CQD, etc.) have been proposed using n -qubit ($n \geq 3$) entanglement. Specifically, various protocols for BCST have been proposed using m -qubit entanglement ($m \in \{5, 6, 7\}$). Here, we propose a PoP-based protocol for BCST that uses only Bell states and thus reduces the complexity of the required quantum resources. We have also provided a Bell-state based protocol of CQD and have studied the effect of amplitude damping and phase damping channels on both the schemes. Further, we have shown that a set of other schemes of controlled quantum communication (e.g., CQSDC, CQKD and CQKA) can be realized solely using Bell states. Extending the discussion, here we note that it is straight forward to realize that any channel that can be used for teleportation can also be used for remote state preparation. Thus, the Bell-state-based realization of BCST discussed above can be easily extended to provide a protocol of CBRSP [40, 43]. Interestingly, in Refs. [40, 43], 5-qubit quantum states are used to implement the same. In Ref. [40], a scheme for controlled bidirectional JRSP (CBJRSP) was proposed using the 7-qubit quantum states of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|GHZ_1\rangle_{123}|GHZ_2\rangle_{456}|a\rangle_7 \pm |GHZ_3\rangle_{123}|GHZ_4\rangle_{456}|b\rangle_7) \quad (12)$$

with $GHZ_1 \neq GHZ_3$ and $GHZ_2 \neq GHZ_4$ and GHZ_i with $i \in \{1, 2, 3, 4\}$ is a GHZ state. Following the same logic as was adopted above to construct a Bell-state based protocol of BCST, we can easily construct a GHZ-based protocol of CBJRSP. In such a protocol, instead of 7-qubit state used in [40], we will only require 3-qubit GHZ states. Clearly, PoP can be used to perform various tasks of controlled quantum communication with entangled states that involve a lesser number of qubits. As it is easier to produce and maintain a Bell state or a GHZ state in comparison to a multi-qubit entangled state, PoP essentially makes it easier to implement the schemes of controlled quantum communication. Further, the strategy used here to design the protocols of controlled quantum communication has a few other advantages over the existing protocols for the same task. For example, we may mention the following advantages: (i) In this protocols, the controller can continuously vary the amount of information revealed to the receiver(s) and the sender(s). (ii) The controller has directional control in the BCST and similar protocols, i.e., he can choose to disclose the information about the Bell state required for the reconstruction of the teleported state in only one direction (say the Bell state used for Alice to Bob teleportation) without disclosing any information about the Bell state used for the teleportation in the other direction (i.e., Bob to Alice teleportation). Such control was not present in earlier schemes. Considering these advantages, we conclude the paper with an expectation that the strategy proposed here will play important role in the future development of the protocols of controlled quantum communication.

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